Lecture – 37.

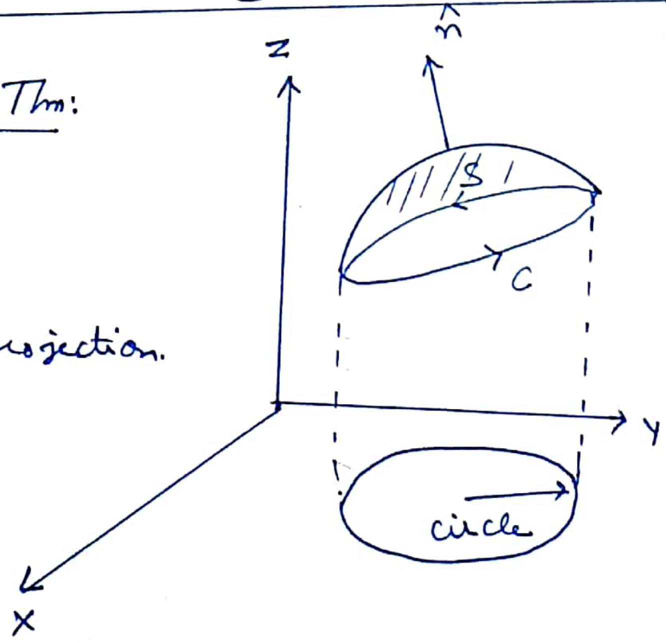
Stoke's Theorem:

Let S be a piecewise smooth orientable surface bounded by piecewise smooth simple closed curve C. Let $\vec{V} = (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$ be a vector function which is continuous and has continuous first order partial derivatives in the domain which contains S. If C is traced in positive direction and $\hat{n}$ is the unit normal vector to S, then

$$\boxed{\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{V})\cdot\hat{n}\, dA}$$

or

$$\boxed{\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{V})\cdot\hat{n}\, dA}$$



To apply Stoke's Thm:

1. Find (curl $\vec{V}$)
2. Calculate $\hat{n}$
3. Find dA by projection.

Use formula.

(2), (III).

Q: Evaluate $\oint_C \vec{V}\cdot d\vec{r}$ using Stoke's theorem

Where $\vec{V} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=16$, $z>0$

Sol:

Here $\vec{V} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$

Let $\vec{r} = (x\hat{i}+y\hat{j}+z\hat{k}) \Rightarrow d\vec{r} = dx\hat{i}+dy\hat{j}+dz\hat{k}$

$$\Rightarrow \vec{V}\cdot d\vec{r} = (3x-y)dx - 2yz^2dy - 2y^2zdz$$

$$\therefore \oint_C \vec{V}\cdot d\vec{r} = \oint_C (3x-y)dx - 2yz^2dy - 2y^2zdz \quad ?$$

Also
$$\text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x-y) & -2yz^2 & -2y^2z \end{vmatrix}$$

$$= i[-4y\cancel{z} + 4yz] - \hat{j}[0-0] + \hat{k}[0+1]$$

$$= 0\hat{i} - 0\hat{j} + 1\hat{k}$$

$$\Rightarrow \boxed{\text{curl}\,\vec{V} = \hat{k}}$$

Let $f = x^2+y^2+z^2 = 16$

$$\Rightarrow \vec{\nabla} f = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$$

$$\Rightarrow \vec{\nabla} f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\Rightarrow \quad \hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2x\hat{i} + 2y\hat{j} + 2z\hat{k})}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}$$

$$= \frac{\cancel{2}(x\hat{i} + y\hat{j} + z\hat{k})}{\cancel{2}\sqrt{x^2 + y^2 + z^2}}$$

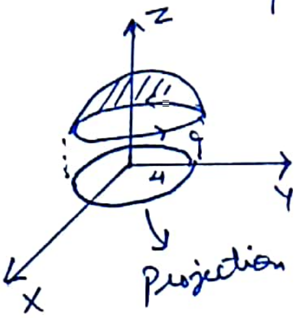
$$= \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{16}}$$

$$\Rightarrow \quad \boxed{\hat{n} = \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{4}}$$

Acc. to Stoke's Thm:

$$\oint_C \vec{V} \cdot d\vec{r} = \iint_S (\text{curl } \vec{V}) \cdot \hat{n} \, dA$$

Let projection be on xy-plane. $(z=0)$



$$\therefore \quad S: x^2 + y^2 + 0 = 16$$

$$\Rightarrow \quad x^2 + y^2 = 16$$

$$\Rightarrow \quad x^2 + y^2 = (4)^2 \quad [\text{circle of radius } 4]$$

Now $$dA = \frac{dx\,dy}{\hat{n} \cdot \hat{k}}$$

$$\Rightarrow \quad dA = \frac{dx\,dy}{\frac{(x\hat{i} + y\hat{j} + z\hat{k})}{4} \cdot \hat{k}}$$

$$\Rightarrow \quad dA = \frac{dx\,dy}{\frac{z}{4}} \quad \Rightarrow \quad \boxed{dA = \frac{4\,dx\,dy}{z}}$$

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(4. [illegible]).

Again, By Stoke's Thm:

$$\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\text{curl } \vec{V})\cdot \hat{n}\, dA$$

$$= \iint_S \left[(\hat{k})\cdot \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{4}\right]\cdot \frac{4\,dx\,dy}{z}$$

$$= \iint_S \frac{\cancel{z}}{4}\cdot \frac{4\,dx\,dy}{\cancel{z}}$$

$$= \frac{\cancel{4}}{\cancel{4}} \iint_S dx\,dy = \iint_S dx\,dy$$

$$= \frac{\cancel{4}}{\cancel{4}}(\text{Area of circle } x^2+y^2=16)$$

$$= \cancel{4}(\pi r^2)$$

$$= \cancel{4}[\pi(4)^2] = 16\pi$$

$$= \cancel{64\pi}$$

$$\Rightarrow \boxed{\oint_C \vec{V}\cdot d\vec{r} = 16\pi} \quad \text{Ans.}$$